

# TORSION IN THE COHOMOLOGY OF TORUS ORBIFOLDS

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**ABSTRACT.** We study torsion in the integral cohomology of a certain family of  $2n$ -dimensional orbifolds  $X$  with actions of the  $n$ -dimensional compact torus. Compact simplicial toric varieties are in our family. For a prime number  $p$ , we find a necessary condition for the integral cohomology of  $X$  to have no  $p$ -torsion. Then we prove that the necessary condition is sufficient in some cases. We also give an example of  $X$  which shows that the necessary condition is not sufficient in general.

## INTRODUCTION

A toric variety is a normal complex algebraic variety of complex dimension  $n$  with an algebraic action of  $(\mathbb{C}^*)^n$  having a dense orbit. A toric variety is not necessarily compact and may have singularity. The famous theorem of Danilov-Jurkiewicz gives an explicit description of the integral cohomology ring of a compact smooth toric variety in terms of the associated fan. It in particular says that the integral cohomology groups are torsion-free and concentrated in even degrees.

The analogous result holds for a compact simplicial toric variety  $X$  (simplicial means that  $X$  is an orbifold) but with rational coefficients. S. Fischli and A. Jordan studied the integral cohomology groups  $H^*(X)$  in their dissertations [7], [11] using spectral sequences. Their results give an explicit computation of  $H^k(X)$  and  $H^{2n-k}(X)$  for  $k \leq 3$  under some conditions. Based on their results, M. Franz developed Maple package `torhom` [8] to compute those cohomology groups. One can see that  $H^*(X)$  has torsion in general while it has no torsion when  $X$  is a weighted projective space ([12]). Therefore we are naturally led to ask when  $H^*(X)$  has torsion or no torsion.

The orbit space  $Q$  of a compact simplicial toric variety  $X$  by the restricted action of the  $n$ -dimensional compact torus  $T$  is a nice manifold with corners (sometimes called a manifold with faces). All faces of  $Q$  (even  $Q$  itself) are contractible and  $Q$  is often homeomorphic to a simple polytope as manifolds with corners. MacPherson showed that  $X$  is homeomorphic to the quotient space  $(Q \times T)/\sim$  under some equivalence relation  $\sim$  defined using the primitive vectors in the one-dimensional cones in the fan of  $X$  (see [9]). The one-dimensional cones correspond to the facets of  $Q$  so that one can think of the primitive vectors as a map

$$v: \{Q_1, Q_2, \dots, Q_m\} \rightarrow \mathbb{Z}^n \quad (Q_i\text{'s are facets of } Q).$$

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The map  $v$  satisfies some linear independence condition and a map satisfying the condition is called a *characteristic function* on  $Q$  (see Definition in Section 1). Note that there are many characteristic functions which do not arise from compact simplicial toric varieties.

Bahri-Sarkar-Song [1] consider the quotient space  $X(Q, v) = (Q \times T)/\sim$ . Although they restrict their concern to  $Q$  being a simple polytope, the characteristic function  $v$  used to define the equivalence relation  $\sim$  is arbitrary; so the quotient space does not necessarily arise from a compact simplicial toric variety. They give a sufficient condition for  $H^*(X(Q, v))$  to be torsion-free in terms of  $Q$  and  $v$ . They also give a Danilov-Jurkiewicz type description for the ring structure of  $H^*(X(Q, v))$  when it is torsion-free.

In this paper, we also consider the quotient space  $X = X(Q, v) = (Q \times T)/\sim$  where  $v$  is arbitrary as above but our  $Q$  is a compact connected nice manifold with corners and not necessarily a simple polytope. When  $Q$  has a vertex (equivalently  $X$  has a  $T$ -fixed point), our  $X$  is a torus orbifold in the sense of [10]. We give an explicit description of  $H^k(X)$  and  $H^{2n-k}(X)$  for  $k \leq 2$  under some condition on  $Q$ . Motivated by the explicit description of  $H^{2n-1}(X)$ , we introduce a positive integer  $\mu(Q_I)$  depending on the characteristic function  $v$  for each  $Q_I = \bigcap_{i \in I} Q_i$ , where  $I$  is a subset of  $\{1, \dots, m\}$  and we understand  $Q_I = Q$  when  $I = \emptyset$  and  $\mu(Q_I) = 1$  when  $Q_I = \emptyset$ . The  $\mu(Q_I)$ 's are all one when  $X$  has no singularity. Here is a summary of our results, which follows from Propositions 5.2, 7.1, 7.2 and 7.4.

**Theorem.** *Let  $Q$  be a connected nice manifold with corners of dimension  $n \geq 1$ . Let  $p$  be a prime number and suppose that every face of  $Q$  (even  $Q$  itself) is acyclic with  $\mathbb{Z}/p$ -coefficients. If  $H^*(X(Q, v))$  has no  $p$ -torsion, then  $\mu(Q_I)$  is coprime to  $p$  for every  $Q_I$ . The converse holds when the face poset of  $Q$  is isomorphic to the face poset of one of the following:*

- (1) *the suspension  $\diamond^n$  of the  $(n-1)$ -simplex  $\Delta^{n-1}$ , i.e.  $\diamond^n$  is obtained from  $\Delta^{n-1} \times [-1, 1]$  by collapsing  $\Delta^{n-1} \times \{1\}$  and  $\Delta^{n-1} \times \{-1\}$  to a point respectively,*
- (2)  $\Delta^n$ ,
- (3)  $\Delta^{n-1} \times [-1, 1]$ .

**Remark.** (1) When  $n \geq 3$ , there are many nice manifolds with corners  $Q$  which have the same face posets as  $\diamond^n$ ,  $\Delta^n$  or  $\Delta^{n-1} \times [-1, 1]$  but not homeomorphic to them. For instance, one can produce such  $Q$  by taking connected sum of them and integral homology  $n$ -spheres with non-trivial fundamental groups.

(2) The  $n$ -simplex  $\Delta^n$  and the prism  $\Delta^{n-1} \times [-1, 1]$  can be obtained from the suspension  $\diamond^n$  by performing a vertex cut once and twice respectively. So, the reader might think that the converse mentioned in the theorem above would hold for  $Q$  obtained from  $\diamond^n$  by performing a vertex cut repeatedly. However, we will see in Section 8 that this is not true for  $Q$  obtained from  $\diamond^3$  by performing a vertex cut four times.

The paper is organized as follows. In Section 1 we set up notations. In Section 2 we compute  $H^{2n-k}(X)$  ( $k \leq 2$ ) for the quotient space  $X = (Q \times T)/\sim$  using the idea in Yeroshkin's paper [17]. Namely, we delete a small neighborhood of the singular set in  $X$  to obtain a smooth manifold and investigate the relation of the cohomology groups between  $X$  and the smooth manifold. In Section 3 we show that the quotient map  $X \rightarrow Q$  induces an isomorphism on their fundamental groups when  $Q$  has a vertex.

In Section 4 we apply the results in Sections 2 and 3 to the case when  $n = 2$  and 3. In Section 5 we introduce  $\mu(Q_I)$  and find a necessary condition for  $H^*(X)$  to have no  $p$ -torsion. In Section 6 we recall Theorem on Elementary Divisors and deduce two facts used in Section 7. In Section 7 we prove that the necessary condition obtained in Section 5 is sufficient for  $Q$  mentioned in the theorem above. Section 8 gives an example mentioned in the remark above. In the appendix we will observe that a result of Fischli or Jordan on  $H^{2n-1}(X)$  and the torsion part of  $H^{2n-2}(X)$  agrees with our Proposition 2.2 when  $X$  is a compact simplicial toric variety.

## 1. SETTING AND NOTATION

In this section, we set up some notations and give some remarks. Let  $Q$  be a connected manifold with corners of dimension  $n$  (see [6, p.180] for the precise definition of a manifold with corners). Then faces are defined and a codimension-one face is called a facet. We assume that  $Q$  is *nice*, which means that every codimension- $k$  face is a connected component of intersections of  $k$  facets. The teardrop, which is homeomorphic to the 2-disk, is a manifold with corners but not nice (see [6, p.181]). A simple polytope is a nice manifold with corners and any intersection of faces is connected unless it is empty. However, intersections of faces of a nice manifold with corners are not necessarily connected. For instance, a 2-gon, that is the suspension  $\diamond^2$  in the theorem in the Introduction, is a nice manifold with corners but the intersection of the two facets consists of two vertices.

Let  $S^1$  be the unit circle group of the complex numbers  $\mathbb{C}$  and  $T$  be an  $n$ -dimensional connected compact abelian Lie group. As is well-known,  $T$  is isomorphic to  $(S^1)^n$ . We set

$$N := \text{Hom}(S^1, T) \cong \mathbb{Z}^n.$$

Let  $Q$  have  $m$  facets and we denote them by  $Q_1, \dots, Q_m$ .

**Definition.** A function  $v: \{Q_1, \dots, Q_m\} \rightarrow N$  is called a *characteristic function on  $Q$*  if it satisfies the following two conditions:

- (1)  $v(Q_i)$  is primitive for each  $i \in [m] := \{1, \dots, m\}$  and
- (2) whenever  $Q_I = \bigcap_{i \in I} Q_i$  is nonempty for  $I \subset [m]$ ,  $v(Q_i)$ 's ( $i \in I$ ) are linearly independent over  $\mathbb{Q}$ .

We denote by  $\hat{N}$  the sublattice of  $N$  generated by  $v_1, \dots, v_m$ .

We call  $v(Q_i)$ 's the *characteristic vectors* and abbreviate  $v(Q_i)$  as  $v_i$ . Condition (2) above implies that when  $Q$  has a vertex,  $\text{rank } \hat{N} = n$ . It also implies that when  $Q_I \neq \emptyset$ , the toral subgroup of  $T$  generated by  $v_i(S^1)$ 's ( $i \in I$ ), denoted by  $T_I$ , is of dimension  $|I|$  where  $|I|$  is the cardinality of  $I$ .

To the pair  $(Q, v)$  we associate a quotient space

$$X(Q, v) := (Q \times T) / \sim$$

with the equivalence relation  $\sim$  on the product  $Q \times T$  defined by

$$(q, t) \sim (q', t') \text{ if and only if } q = q' \text{ and } t^{-1}t' \in T_I$$

where  $I$  is the subset of  $[m]$  such that  $Q_I$  is the smallest face of  $Q$  containing  $q = q'$ . The space  $X(Q, v)$  has a  $T$ -action induced from the natural  $T$ -action on  $Q \times T$ . The orbit space of  $X(Q, v)$  by the  $T$ -action is  $Q$  and the quotient map

$$\pi: X(Q, v) \rightarrow Q = X(Q, v) / T$$

is induced from the projection map  $Q \times T \rightarrow Q$ . Then it is not difficult to see the following facts (see [13] for example). A  $T$ -fixed point in  $X(Q, v)$  corresponds to a vertex of  $Q$ , so  $X(Q, v)$  has a  $T$ -fixed point if and only if  $Q$  has a vertex. If  $v_i$ 's ( $i \in I$ ) are a part of a basis of  $N$  for every  $I$  with  $Q_I \neq \emptyset$ , then  $X(Q, v)$  is a manifold but otherwise  $X(Q, v)$  is an orbifold. The singularity of  $X(Q, v)$  lies in the union of  $\pi^{-1}(Q_I)$  over all  $I$  with  $|I| \geq 2$ .

As mentioned in the Introduction, if  $X$  is a compact simplicial toric variety of complex dimension  $n$  so that  $X$  has an algebraic action of  $(\mathbb{C}^*)^n$  having a dense orbit, then the orbit space  $Q$  of  $X$  by the compact  $n$ -dimensional subtorus  $T$  of  $(\mathbb{C}^*)^n$  is a nice manifold with corners and  $X$  is homeomorphic to  $X(Q, v)$  where  $v_i$ 's are primitive edge vectors of the fan associated to  $X$ . Moreover, faces of  $Q$  (even  $Q$  itself) are all contractible, which follows from the existence of the residual action of  $(\mathbb{C}^*)^n/T$  on  $Q = X/T$ .

## 2. $H^{2n-k}(X(Q, v))$ FOR $k \leq 2$

In this section, we abbreviate  $X(Q, v)$  as  $X$  and all (co)homology groups will be taken with  $\mathbb{Z}$ -coefficients unless otherwise stated. When  $n = 1$ ,  $Q$  is a closed interval if  $Q$  has a vertex and a circle otherwise, and  $X$  is homeomorphic to  $S^2$  or a torus accordingly. We will assume  $n \geq 2$  in this section. Remember that  $\pi: X \rightarrow Q$  is the quotient map.

Let  $Q^{(n-2)}$  be the union of  $Q_I$  over all  $I$  with  $|I| \geq 2$  and we assume  $Q^{(n-2)} \neq \emptyset$ . The singular set of  $X$  lies in  $\pi^{-1}(Q^{(n-2)})$  as remarked in Section 1. Let  $Q'$  be a “small closed tubular neighborhood” of  $Q^{(n-2)}$  of  $Q$  and set  $X' := \pi^{-1}(Q')$ .

**Lemma 2.1.**  $H^{2n-k}(X) \cong H_k(X \setminus \text{Int } X')$  for  $k \leq 2$ .

*Proof.* Note that  $H^r(X') = 0$  for  $r \geq 2n - 3$  because  $X'$  is homotopy equivalent to  $\pi^{-1}(Q^{(n-2)})$  and  $\dim \pi^{-1}(Q^{(n-2)}) = 2n - 4$ . Therefore, the exact sequence in cohomology for the pair  $(X, X')$  yields an isomorphism

$$(2.1) \quad H^{2n-k}(X, X') \cong H^{2n-k}(X) \quad \text{for } k \leq 2.$$

On the other hand,

$$(2.2) \quad \begin{aligned} H^{2n-k}(X, X') &\cong H^{2n-k}(X \setminus \text{Int } X', \partial X') \quad \text{by excision} \\ &\cong H_k(X \setminus \text{Int } X') \quad \text{by Poincaré-Lefschetz duality.} \end{aligned}$$

(Note that  $X \setminus \text{Int } X'$  is a manifold with boundary  $\partial X'$ .) The lemma follows from (2.1) and (2.2).  $\square$

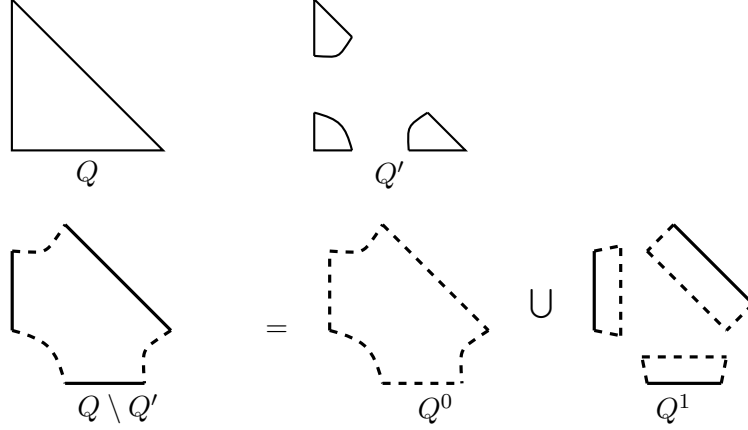
**Proposition 2.2.**  $H^{2n}(X) \cong \mathbb{Z}$  and  $H^{2n-1}(X) \cong H_1(Q) \oplus N/\hat{N}$ . If  $H_1(Q_i) = 0$  for every  $i$ , then

$$H^{2n-2}(X) \cong \mathbb{Z}^{m-\text{rank } \hat{N}} \oplus H_2(Q) \oplus (H_1(Q) \otimes H_1(T)) \oplus (\wedge^2 N/\hat{N} \wedge N).$$

**Remark.** When  $Q$  has a vertex,  $\text{rank } \hat{N} = n$  as remarked in Section 1. Moreover, when  $Q$  has a vertex and  $n = 2$ , the last term  $\wedge^2 N/\hat{N} \wedge N$  above is zero. Indeed, since we may assume  $N = \mathbb{Z}^2$  and  $\hat{N} = \langle e_1, ae_2 \rangle$  with some integer  $a$ ,  $\hat{N} \wedge N = \langle e_1 \wedge e_2 \rangle = \wedge^2 N$ , where  $\{e_1, e_2\}$  denotes the standard base of  $\mathbb{Z}^2$ .

*Proof.* The statement for  $H^{2n}(X)$  follows immediately from Lemma 2.1.

We shall prove the statement for  $H^{2n-1}(X)$ . Let  $Q^0 := (\text{Int } Q) \cap (Q \setminus Q')$  and  $Q^1$  be the intersection of  $(Q \setminus Q')$  and a small open neighborhood of  $\partial Q$  in  $Q$ .



Since

$$\begin{aligned} \pi^{-1}(Q^0) &\simeq Q \times T, \quad \pi^{-1}(Q^1) \simeq \bigsqcup_{i=1}^m (Q_i \times T/v_i(S^1)), \\ \pi^{-1}(Q^0) \cap \pi^{-1}(Q^1) &\simeq \bigsqcup_{i=1}^m (Q_i \times T), \quad \pi^{-1}(Q^0 \cup Q^1) = X \setminus X', \end{aligned}$$

the Mayer-Vietoris exact sequence in homology for the triple  $(X \setminus X', \pi^{-1}(Q^0), \pi^{-1}(Q^1))$  yields the following exact sequence:

$$\begin{aligned} (2.3) \quad & \bigoplus_{i=1}^m H_2(Q_i \times T) \xrightarrow{f_2} H_2(Q \times T) \oplus \bigoplus_{i=1}^m H_2(Q_i \times T/v_i(S^1)) \rightarrow H_2(X \setminus X') \\ & \rightarrow \bigoplus_{i=1}^m H_1(Q_i \times T) \xrightarrow{f_1} H_1(Q \times T) \oplus \bigoplus_{i=1}^m H_1(Q_i \times T/v_i(S^1)) \rightarrow H_1(X \setminus X') \\ & \rightarrow \bigoplus_{i=1}^m H_0(Q_i \times T) \xrightarrow{f_0} H_0(Q \times T) \oplus \bigoplus_{i=1}^m H_0(Q_i \times T/v_i(S^1)). \end{aligned}$$

As is easily seen,  $f_0$  is injective; so

$$(2.4) \quad H_1(X \setminus X') \cong \text{coker } f_1.$$

We write  $f_1$  as  $(\psi_1, \varphi_1)$  according to the decomposition of the target space. Since

$$\varphi_1: \bigoplus_{i=1}^m H_1(Q_i \times T) \rightarrow \bigoplus_{i=1}^m H_1(Q_i \times T/v_i(S^1)),$$

which is  $f_1$  composed with the projection on the second factor, is surjective, one has

$$(2.5) \quad \text{coker } f_1 \cong H_1(Q \times T)/\psi_1(\ker \varphi_1).$$

Since  $H_1(Y \times T) = H_1(Y) \oplus H_1(T)$  for any topological space  $Y$ , elements in  $\ker \varphi_1$  are of the form  $(c_1 v_1, \dots, c_m v_m)$  with integers  $c_i$ , where  $H_1(T)$  is identified with

$N = \text{Hom}(S^1, T)$  in a natural way. It follows that

$$(2.6) \quad H_1(Q \times T)/\psi_1(\ker \varphi_1) \cong H_1(Q) \oplus N/\hat{N}.$$

The statement for  $H^{2n-1}(X)$  in the proposition follows from (2.4), (2.5), (2.6) and Lemma 2.1.

The computation of  $H^{2n-2}(X)$  is similar to that of  $H^{2n-1}(X)$ . We write  $f_2$  as  $(\psi_2, \varphi_2)$  similarly to  $f_1$ . Since  $H_1(Q_i) = 0$  for any  $i$  by assumption,  $\ker f_1$  is a free abelian group of rank  $m - \text{rank } \hat{N}$  as is easily seen; so it follows from (2.3) that

$$(2.7) \quad H_2(X \setminus X') \cong \mathbb{Z}^{m - \text{rank } \hat{N}} \oplus \text{coker } f_2.$$

Similarly to  $\varphi_1$ , the map

$$(2.8) \quad \varphi_2: \bigoplus_{i=1}^m H_2(Q_i \times T) \rightarrow \bigoplus_{i=1}^m H_2(Q_i \times T/v_i(S^1))$$

is surjective; so

$$(2.9) \quad \text{coker } f_2 \cong H_2(Q \times T)/\psi_2(\ker \varphi_2).$$

Here,

$$(2.10) \quad H_2(Y \times T) = H_2(Y) \oplus (H_1(Y) \otimes H_1(T)) \oplus H_2(T)$$

for any topological space  $Y$  by the Künneth formula. Therefore, since  $H_1(Q_i) = 0$  by assumption, it follows from (2.8) and (2.10) that  $\ker \varphi_2$  is contained in  $\bigoplus_{i=1}^m H_2(T)$ . We note that  $H_2(T)$  and  $H_2(T/v_i(S^1))$  can be identified with  $\wedge^2 N$  and  $\wedge^2(N/\langle v_i \rangle)$  respectively and the kernel of the projection  $\wedge^2 N \rightarrow \wedge^2(N/\langle v_i \rangle)$  is  $\langle v_i \rangle \wedge N$ . Therefore

$$\text{coker } f_2 \cong H_2(Q) \oplus (H_1(Q) \otimes H_1(T)) \oplus (\wedge^2 N/\hat{N} \wedge N)$$

This together with (2.7) and (2.9) proves the statement for  $H^{2n-2}(X)$  in the proposition.  $\square$

### 3. FUNDAMENTAL GROUPS

For a subset  $I$  of  $[m]$ , we define

$$T_I^m := \{(h_1, \dots, h_m) \in T^m \mid h_j = 1 \quad (\forall j \notin I)\}.$$

and consider a space

$$\mathcal{Z}_Q := (Q \times T^m)/\sim_e$$

where  $\sim_e$  is the equivalence relation on the product  $Q \times T^m$  defined by

$$(q, s) \sim_e (q', s') \text{ if and only if } q = q' \text{ and } s^{-1}s' \in T_I^m$$

and  $I$  is the subset of  $[m]$  such that  $Q_I$  is the smallest face of  $Q$  containing  $q = q'$ .

We note that  $\mathcal{Z}_Q$  locally admits a smooth structure. Indeed, since  $Q$  is a manifold with corners, any point of  $Q$  has a neighborhood  $U$  homeomorphic to  $(\mathbb{R}_{\geq 0})^r \times \mathbb{R}^{n-r}$  for some  $0 \leq r \leq n$  and it follows from the construction of  $\mathcal{Z}_Q$  that the inverse image of  $U$  by the projection map  $\kappa: \mathcal{Z}_Q \rightarrow Q$  is homeomorphic to  $\mathbb{C}^r \times \mathbb{R}^{n-r} \times T^{m-r}$ . Therefore  $\mathcal{Z}_Q$  locally admits a smooth structure and hence is a topological manifold.

**Remark.** When  $Q$  is a simple polytope,  $\mathcal{Z}_Q$  is called a moment-angle manifold and it is known that  $\mathcal{Z}_Q$  admits a smooth structure and is 2-connected (see [3] or [4]). Moreover, the moment-angle manifold  $\mathcal{Z}_Q$  is homotopy equivalent to  $\mathbb{C}^m - Z$  defined in [5] (see Theorem 4.7.5 in [4]), where  $Z$  is the union of coordinate subspaces in  $\mathbb{C}^m$  determined by  $Q$ .

**Lemma 3.1.** *The projection map  $\kappa: \mathcal{Z}_Q \rightarrow Q$  induces an isomorphism  $\kappa_*: \pi_1(\mathcal{Z}_Q) \cong \pi_1(Q)$  on the fundamental groups.*

*Proof.* Similarly to the above argument, one can see that  $\kappa^{-1}(Q_i)$ , where  $Q_i$  is a facet of  $Q$ , is a locally smooth closed manifold. Moreover, it is a locally smooth codimension two submanifold of  $\mathcal{Z}_Q$ . Indeed, a closed tubular neighborhood of  $Q_i$  in  $Q$  can be identified with  $Q_i \times [0, 1]$ , and  $\rho_i: \kappa^{-1}(Q_i \times \{1\}) \rightarrow \kappa^{-1}(Q_i)$ , where  $\rho_i$  is induced from  $((q, 1), t) \rightarrow (q, t)$  for  $q \in Q_i = Q_i \times \{0\} \subset Q_i \times [0, 1] \subset Q$  and  $t \in T^m$ , is a principal  $S^1$ -bundle, and the total space  $E_i$  of the associated complex line bundle can be identified with a closed tubular neighborhood of  $Z_i := \kappa^{-1}(Q_i)$  in  $\mathcal{Z}_Q$ .

Since  $Z_i$  is a locally smooth closed codimension two submanifold of  $\mathcal{Z}_Q$ , the transversality argument can be applied. Therefore, if a continuous map  $f: S^1 \rightarrow \mathcal{Z}_Q$  meets  $Z_i$ , then one can slightly push  $f$  in the fiber direction of  $E_i$  so that the deformed  $f$  does not meet  $Z_i$ . Applying this deformation to  $f$  for every  $i$ , we see that  $f$  is homotopic to a continuous map whose image lies in  $\kappa^{-1}(\text{Int } Q) = \text{Int } Q \times T^m$ . This means that the inclusion map  $\iota: \text{Int } Q \times T^m \rightarrow \mathcal{Z}_Q$  induces an epimorphism

$$\iota_*: \pi_1(\text{Int } Q \times T^m) = \pi_1(\text{Int } Q) \times \pi_1(T^m) \rightarrow \pi_1(\mathcal{Z}_Q).$$

Since  $\text{Int } Q$  is homotopy equivalent to  $Q$ , we may replace  $\text{Int } Q$  by  $Q$  above and we have a sequence

$$(3.1) \quad \pi_1(Q) \times \pi_1(T^m) \xrightarrow{\iota_*} \pi_1(\mathcal{Z}_Q) \xrightarrow{\kappa_*} \pi_1(Q),$$

where the composition  $\kappa_* \circ \iota_*$  agrees with the projection on the first factor, so that the kernel of  $\iota_*$  is contained in the second factor  $\pi_1(T^m)$ .

Let  $S_i$  be the  $i$ -th  $S^1$ -factor of  $T^m$  and choose a point  $q_i \in (Q_i \times \{1\}) \cap \text{Int } Q$ . Then  $\iota(\{q_i\} \times S_i)$  is a fiber of the principal  $S^1$ -bundle  $\rho_i: \kappa^{-1}(Q_i \times \{1\}) \rightarrow Z_i = \kappa^{-1}(Q_i)$ , so it shrinks to a point in  $Z_i$ . Therefore  $\pi_1(T^m)$  is in the kernel of the epimorphism  $\iota_*$  and this implies the lemma.  $\square$

We recall a result from Bredon's book [2].

**Lemma 3.2.** [2, Corollary 6.3 on p.91]. *If  $X$  is an arcwise connected  $G$ -space,  $G$  compact Lie, and if there is an orbit which is connected (e.g.,  $G$  connected or  $X^G \neq \emptyset$ ), then the quotient map  $X \rightarrow X/G$  induces an epimorphism on their fundamental groups.*

The characteristic map  $v: \{Q_1, \dots, Q_m\} \rightarrow \text{Hom}(S^1, T)$  defines a homomorphism  $T^m \rightarrow T$ , denoted  $v$  again. Note that  $v(T^m)$  is a subtorus of  $T$  of dimension  $\text{rank } \hat{N}$ , in particular,  $v$  is surjective if and only if  $\text{rank } \hat{N} = \text{rank } N$  (this is the case when  $Q$  has a vertex). The product map  $\text{id} \times v: Q \times T^m \rightarrow Q \times T$  induces a continuous map

$$V: \mathcal{Z}_Q = Q \times T^m / \sim_e \rightarrow Q \times T / \sim = X(Q, v) = X$$

and it further induces an injective continuous map

$$\bar{V}: \mathcal{Z}_Q / \ker v \rightarrow X,$$

so that  $\bar{V}$  is a homeomorphism if  $v$  is surjective since the spaces are compact and Hausdorff.

**Proposition 3.3.** *If  $Q$  has a vertex, then  $\pi_*: \pi_1(X) \cong \pi_1(Q)$ .*

*Proof.* We have a sequence

$$\kappa_* = \pi_* \circ V_*: \pi_1(\mathcal{Z}_Q) \xrightarrow{V_*} \pi_1(X) \xrightarrow{\pi_*} \pi_1(Q).$$

Since  $\kappa_*$  is an isomorphism by Lemma 3.1, it suffices to prove that  $V_*$  is surjective.

Since  $Q$  has a vertex,  $\text{rank } \hat{N} = \text{rank } N$  and the homomorphism  $v: T^m \rightarrow T$  is surjective; so the map  $\tilde{V}: \mathcal{Z}_Q / \ker v \rightarrow X$  above is a homeomorphism. Since  $\hat{N}$  is a sublattice of  $N$  of finite index, there is a finite covering homomorphism  $\rho: \hat{T} \rightarrow T$  corresponding to  $\hat{N}$ , where  $\hat{T}$  is also a compact connected abelian Lie group of dimension  $n$  (precisely speaking,  $\rho_*(\pi_1(\hat{T})) = \hat{N}$  when  $N$  is regarded as  $\pi_1(T)$ ) and the characteristic function  $v$  uniquely determines a characteristic function  $\hat{v}: \{Q_1, \dots, Q_m\} \rightarrow \text{Hom}(S^1, \hat{T})$  such that  $\rho_*(\hat{v}(Q_i)) = v(Q_i)$  for any  $i$ . Then we have

$$\hat{X} := X(Q, \hat{v}) = (Q \times \hat{T}) / \sim$$

and  $\hat{v}$  induces a homomorphism  $T^m \rightarrow \hat{T}$ , denoted  $\hat{v}$  again similarly to  $v$ , and  $\hat{X} = \mathcal{Z}_Q / \ker \hat{v}$ . Moreover, we have  $X = \hat{X} / \ker \rho$ . Namely, the quotient map  $V: \mathcal{Z}_Q \rightarrow X$  factors as the composition of two quotient maps

$$\mathcal{Z}_Q \xrightarrow{\alpha} \mathcal{Z}_Q / \ker \hat{v} = \hat{X} \xrightarrow{\beta} \hat{X} / \ker \rho = X.$$

The Theorem on Elementary Divisors (see Section 6) implies that since  $\hat{v}(Q_i)$ 's span  $\hat{N}$ , the homomorphism  $\hat{v}: T^m \rightarrow \hat{T}$  composed with a suitable automorphism of  $T^m$  can be viewed as a projection map if we take a suitable identification of  $\hat{T}$  with  $T^n$ ; so  $\ker \hat{v}$  is connected and hence  $\alpha_*: \pi_1(\mathcal{Z}_Q) \rightarrow \pi_1(\hat{X})$  is surjective by Lemma 3.2. The action of  $\hat{T}$  on  $\hat{X}$  has a fixed point since  $Q$  has a vertex and  $\ker \rho$  is contained in  $\hat{T}$ , so the action of  $\ker \rho$  on  $\hat{X}$  has a fixed point. Therefore  $\beta_*: \pi_1(\hat{X}) \rightarrow \pi_1(X)$  is also surjective again by Lemma 3.2.  $\square$

**Remark.** As mentioned in the Introduction, even if  $Q$  is a simple polytope,  $X = \mathcal{Z}_Q / \ker v$  is not necessarily a compact toric orbifold because the characteristic map  $v$  is not necessarily coming from primitive vectors of a complete simplicial fan.

**Corollary 3.4.** *If  $Q$  has a vertex and  $H_1(Q) = H_2(Q) = 0$ , then  $H^1(X) = 0$  and  $H^2(X) \cong \mathbb{Z}^{m-n}$ .*

*Proof.* By Proposition 3.3,  $\pi_1(X) \cong \pi_1(Q)$  and hence  $H_1(X) \cong H_1(Q)$ . Therefore  $H_1(X) = 0$  since  $H_1(Q) = 0$  by assumption and hence  $H^1(X) = 0$  and  $H^2(X)$  has no torsion by the universal coefficient theorem. On the other hand, since  $X$  is an orbifold, Poincaré duality holds with  $\mathbb{Q}$ -coefficients. Therefore the rank of  $H^2(X)$  is equal to that of  $H^{2n-2}(X)$ , that is  $m - n$  by Proposition 2.2 and its subsequent remark.  $\square$

#### 4. LOW DIMENSIONAL CASES

A nice manifold with corners  $Q$  is called *face-acyclic* ([14]) if every face of  $Q$  (even  $Q$  itself) is acyclic. We note that if  $Q$  is face-acyclic, then  $Q$  must have a vertex. Indeed, let  $F$  be a face of  $Q$  of minimum dimension. Then  $F$  has no boundary because the boundary of  $F$  must consist of faces of smaller dimensions, so  $F$  is a closed manifold. But since  $F$  is acyclic, this means that  $F$  is a point. Therefore  $Q$  has a vertex.

We shall apply the previous results when  $Q$  is face-acyclic and  $n = \dim Q$  is 2 or 3. The following corollary follows from Proposition 2.2 and Corollary 3.4.



**Corollary 4.1.** *Suppose that  $Q$  is face-acyclic and  $\dim Q = 2$ , that is,  $Q$  is an  $m$ -gon ( $m \geq 2$ ). Then we have*

$$H^j(X) \cong \begin{cases} \mathbb{Z} & (j = 0, 4) \\ \mathbb{Z}^{m-2} & (j = 2) \\ N/\hat{N} & (j = 3) \\ 0 & (\text{otherwise}). \end{cases}$$

**Example.** Let  $a$  be a positive integer. Take  $Q$  to be a 2-simplex,  $N = \mathbb{Z}^2$  and

$$v_1 = (2a, 1), \quad v_2 = (0, 1), \quad v_3 = (-a, -1).$$

Then  $\hat{N} = \langle ae_1, e_2 \rangle$  and  $N/\hat{N} \cong \mathbb{Z}/a$ . The space  $X$  is not a weighted projective space when  $a \geq 2$  since it has torsion in cohomology, where  $\{e_1, e_2\}$  denotes the standard base of  $\mathbb{Z}^2$  as before.

**Corollary 4.2.** *Suppose that  $Q$  is face-acyclic and  $\dim Q = 3$ . Then*

$$H^j(X) \cong \begin{cases} \mathbb{Z} & (j = 0, 6) \\ \mathbb{Z}^{m-3} & (j = 2) \\ 0 \text{ or some torsion group} & (j = 3) \\ \mathbb{Z}^{m-3} \oplus \wedge^2 N / (\hat{N} \wedge N) & (j = 4) \\ N/\hat{N} & (j = 5) \\ 0 & (\text{otherwise}). \end{cases}$$

*Proof.* Since  $Q$  is face-acyclic,  $Q$  has a vertex as remarked at the beginning of this section; so all the statements except for  $j = 3$  follows from Proposition 2.2 and Corollary 3.4. In order to prove the statement for  $j = 3$ , it suffices to show  $H^3(X; \mathbb{Q}) = 0$  and this is equivalent to showing that the euler characteristic of  $X$  is  $2m - 4$  (note that we know the rank of  $H^j(X)$  except for  $j = 3$ ).

Since  $Q$  is face-acyclic and of dimension 3, the boundary of  $Q$  is a 2-sphere, every 2-face of  $Q$  is a 2-disk and the number of 2-faces is  $m$  by definition. Let  $V$  be the number of vertices of  $Q$ . Then the number of edges of  $Q$  is  $3V/2$  and hence we obtain an identity  $V - 3V/2 + m = 2$  by Euler's formula, which implies  $V = 2m - 4$ . On the other hand, it is known that the euler characteristic of  $X$  is equal to that of the  $T$ -fixed point set  $X^T$  (see [2, Theorem 10.9 in p.163]). In our case  $X^T$  is isolated and corresponds to the vertices of  $Q$ . Therefore, the euler characteristic of  $X$  is equal to  $V$ , that is  $2m - 4$ .  $\square$

**Example.** It happens that  $\hat{N} \wedge N = \wedge^2 N$  even if  $\hat{N} \neq N$ . For instance, take  $Q$  to be a 3-simplex,  $N = \mathbb{Z}^3$  and

$$v_1 = (0, 0, 1), \quad v_2 = (2, 0, 1), \quad v_3 = (0, 1, 1), \quad v_4 = (-2, -1, -1).$$

Then

$$\hat{N} = \langle 2e_1, e_2, e_3 \rangle, \quad \hat{N} \wedge N = \langle e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3 \rangle = \wedge^2 N,$$

where  $\{e_1, e_2, e_3\}$  denotes the standard base of  $\mathbb{Z}^3$ .

Corollary 4.2 says that if  $\hat{N} = N$ , then  $H^j(X)$  has no torsion except  $j = 3$ . However,  $H^3(X)$  can be nontrivial (so, a nontrivial torsion group) when  $\hat{N} = N$ . We shall give such an example below. One can also find many such examples using Maple package `torhom`.

**Example.** Let  $a$  be a positive integer and take the following five primitive vectors in  $\mathbb{Z}^3$ :

$$\begin{aligned} v_+ &= (0, 0, 1), \\ v_1 &= (2a, 1, 0), \quad v_2 = (0, 1, 0), \quad v_3 = (-a, -1, 0), \\ v_- &= (1, 0, -1). \end{aligned}$$

Then  $\hat{N} = N$ . We consider the complete simplicial fan  $\Delta$  having the following six 3-dimensional cones

$$\angle v_+ v_1 v_2, \angle v_+ v_1 v_3, \angle v_+ v_2 v_3, \angle v_- v_1 v_2, \angle v_- v_1 v_3, \angle v_- v_2 v_3$$

where  $\angle v_\epsilon v_i v_j$  ( $\epsilon \in \{+, -\}$ ,  $i, j \in \{1, 2, 3\}$ ) denotes the cone spanned by  $v_\epsilon, v_i$  and  $v_j$ . Let  $X$  be the compact simplicial toric variety associated to the fan  $\Delta$ . Let  $\rho$  be the projection of  $\mathbb{R}^3$  on the line  $\mathbb{R}$  corresponding to the last coordinates of  $\mathbb{R}^3$ . Then the vectors  $v_1, v_2, v_3$  are in the kernel of  $\rho$  and  $\rho(v_\pm)$  are primitive vectors and determine the complete 1-dimensional fan. This means that we have a fibration  $F \rightarrow X \rightarrow \mathbb{CP}^1$  where the fiber  $F$  is the compact simplicial toric variety associated to the fan obtained by projecting the fan  $\Delta$  on the plane  $\mathbb{R}^2$  corresponding to the first two coordinates of  $\mathbb{R}^3$ . The  $E_2$ -terms of the Serre spectral sequence of the fibration are

$$E_2^{p,q} = H^p(\mathbb{CP}^1; H^q(F))$$

and  $E_2^{p,q} = 0$  unless  $p = 0, 2$  and  $q = 0, 2, 3, 4$  by Corollary 4.1. Therefore all the differentials except

$$d_2^{0,3}: E_2^{0,3} \rightarrow E_2^{2,2} \quad \text{and} \quad d_2^{0,4}: E_2^{0,4} \rightarrow E_2^{2,3}$$

are trivial. Here,  $E_2^{0,3} = H^0(\mathbb{CP}^1; H^3(F)) = H^3(F)$  is trivial or a torsion group by Corollary 4.1 while  $E_2^{2,2} = H^2(\mathbb{CP}^1; H^2(F)) = H^2(F)$  is a free abelian group again by Corollary 4.1, so  $d_2^{0,3}$  must be trivial. Therefore  $E_2^{0,3} = E_\infty^{0,3}$ . Since  $E_2^{p,q}$  with  $p + q = 3$  vanishes unless  $(p, q) = (0, 3)$ , we obtain an isomorphism  $H^3(X) \cong H^3(F)$ . Here  $H^3(F) \cong \mathbb{Z}/a$  again by Corollary 4.1 (see Example after Corollary 4.1) and hence we have  $H^3(X) \cong \mathbb{Z}/a$ . On the other hand, since  $\hat{N} = N$  as remarked above,  $H^j(X)$  has no torsion for  $j \neq 3$  by Corollary 4.2.

## 5. A NECESSARY CONDITION FOR NO $p$ -TORSION

Let  $I$  be a subset of  $[m]$  with  $Q_I \neq \emptyset$ . Although  $Q_I$  is not necessarily connected, we understand that  $Q_I$  stands for a connected component of  $Q_I$  in this section for notational convenience. Then the characteristic function  $v$  associates a characteristic function  $v_I$  on  $Q_I$  as follows. Since  $v_i$ 's ( $i \in I$ ) are linearly independent over  $\mathbb{Q}$ , they span a  $|I|$ -dimensional linear subspace of  $N \otimes \mathbb{R}$  and its intersection with  $N$  is a rank  $|I|$  sublattice of  $N$ , denoted  $N_I$ . Then  $N(I) := N/N_I$  is a free abelian group of rank  $n - |I|$  and we denote the projection map from  $N$  to  $N(I)$  by  $\pi_I$ . If  $Q_I \cap Q_j$  is nonempty for  $j \in [m] \setminus I$ , then its connected components are facets of  $Q_I$ , and any facet of  $Q_I$  is of this form. The element  $\pi_I(v_j) \in N(I)$  is not necessarily primitive and we define  $v_I(Q_I \cap Q_j)$  to be the primitive vector in  $N(I)$  which has the same direction as  $\pi_I(v_j)$ , where  $Q_I \cap Q_j$  also stands for a connected component of  $Q_I \cap Q_j$ . Then one can see that  $v_I$  is a characteristic function on  $Q_I$ . Similarly to  $\hat{N}$ , one can define a sublattice  $\hat{N}(I)$  of  $N(I)$  using  $v_I$ . We allow  $I = \emptyset$  and understand  $Q_\emptyset = Q$ ,

$N(\emptyset) = N$  and  $\hat{N}(\emptyset) = \hat{N}$ . We define

$$\mu(Q_I) := \begin{cases} |N(I)/\hat{N}(I)| & \text{when } Q_I \neq \emptyset, \\ 1 & \text{when } Q_I = \emptyset. \end{cases}$$

Here  $|N(I)/\hat{N}(I)|$  is not necessarily finite. For instance, take  $Q = S^1 \times [-1, 1]$  and assign characteristic vectors  $(1, 0)$  and  $(-1, 0)$  to the facets  $S^1 \times \{1\}$  and  $S^1 \times \{-1\}$  respectively. Then  $N/\hat{N}$  is an infinite cyclic group and hence  $|N(I)/\hat{N}(I)|$  is infinite for  $I = \emptyset$ . One can easily construct a similar example such that  $|N(I)/\hat{N}(I)|$  is infinite for some  $I \neq \emptyset$ .

**Remark.** When  $|I| = n$ ,  $N(I) = \{0\}$ ; so  $\mu(Q_I) = 1$ . When  $|I| = n - 1$ ,  $N(I)$  is of rank one and  $\hat{N}(I)$  is generated by a primitive vector; so  $\hat{N}(I) = N(I)$  and hence  $\mu(Q_I) = 1$  in this case too. Another case which ensures  $\mu(Q_I) = 1$  is the following. Let  $q$  be a vertex of  $Q$ . Then there is a subset  $J$  of  $[m]$  with  $|J| = n$  such that  $q \in Q_J$ . If  $\{v_j\}_{j \in J}$  is a base of  $N$ , then  $\mu(Q_I) = 1$  for every subset  $I$  of  $J$ , which easily follows from the definition of  $\mu(Q_I)$ .

We note that for a prime number  $p$ ,  $H^*(X(Q, v); \mathbb{Z})$  has no  $p$ -torsion if and only if  $H^{\text{odd}}(X(Q, v); \mathbb{Z}/p) = 0$ , which follows from the universal coefficient theorem (see [15, Corollary 56.4]).

**Lemma 5.1.** [2, Theorem 2.2 on pp.376-377]. *Let a group  $G$  of prime order  $p$  act on a finite dimensional space  $X$  with  $A \subset X$  closed and invariant. Suppose that  $G$  acts trivially on  $H^*(X, A; \mathbb{Z})$ . Then*

$$\sum_{i \geq 0} \text{rk } H^{k+2i}(X^G, X^G \cap A; \mathbb{Z}/p) \leq \sum_{i \geq 0} \text{rk } H^{k+2i}(X, A; \mathbb{Z}/p).$$

**Proposition 5.2.** *If  $H^{\text{odd}}(X(Q, v); \mathbb{Z}/p) = 0$ , then  $H_1(Q_I; \mathbb{Z}/p) = 0$  and  $\mu(Q_I)$  is finite and coprime to  $p$  for every  $I$ .*

*Proof.* We abbreviate  $X(Q, v)$  as  $X$  as before. Since  $H^{\text{odd}}(X; \mathbb{Z}/p) = 0$ , we have  $H^{\text{odd}}(X^G; \mathbb{Z}/p) = 0$  for every  $p$ -subgroup  $G$  of  $T_I$  by repeated use of Lemma 5.1. In fact, let  $G$  be an order  $p$  subgroup of  $S^1$ . The induced action of  $G$  on  $H^*(X)$  is trivial because  $G$  is contained in the connected group  $S^1$ . Then  $\text{rk } H^{\text{odd}}(X^G; \mathbb{Z}/p) \leq \text{rk } H^{\text{odd}}(X; \mathbb{Z}/p)$  by Lemma 5.1 applied with  $A = \emptyset$ . Therefore,  $H^{\text{odd}}(X^G; \mathbb{Z}/p) = 0$  by assumption. Repeating the same argument for  $X^G$  with the induced action of  $S^1/G$ , which is again a circle group, we conclude that  $H^{\text{odd}}(X^G; \mathbb{Z}/p) = 0$  for any  $p$ -subgroup  $G$  of  $S^1$ .

For a positive integer  $k$ , let  $G_k$  be the  $p$ -subgroup of  $T_I$  consisting of all elements of order at most  $p^k$ . Then  $G_k \subset G_{k'}$  for  $k \leq k'$  and the union  $\bigcup_{k=1}^{\infty} G_k$  is dense in  $T_I$ . Therefore  $X^{G_k} = X^{T_I}$  if  $k$  is sufficiently large.<sup>1</sup> Since  $X_I = \pi^{-1}(Q_I)$  is a connected component of  $X^{T_I}$ , this shows that  $H^{\text{odd}}(X_I; \mathbb{Z}/p) = 0$ . But  $H^{2(n-|I|)-1}(X_I)$  is isomorphic to  $H_1(Q_I) \oplus N(I)/\hat{N}(I)$  by Proposition 2.2 and hence the universal coefficient theorem implies the proposition.  $\square$

<sup>1</sup>Detailed explanation about this assertion. Since the set of isotropy groups of  $X$  is finite, there is a positive integer  $r$  such that  $X^{G_k} = X^{G_r}$  for every  $k \geq r$ . Since  $G_r$  is a subgroup of  $T_I$ , we have  $X^{G_r} \supset X^{T_I}$ . We shall prove the opposite inclusion. Let  $x \in X^{G_r}$ . The isotropy subgroup  $T_x$  at  $x$  contains  $G_k$  for every  $k \geq r$  because  $X^{G_k} = X^{G_r}$  but since  $T_x$  is a closed subgroup of  $T$ ,  $T_x$  must contain the closure of  $\bigcup_{k=r}^{\infty} G_k$ , that is  $T_I$ . Therefore  $x \in X^{T_I}$  and hence  $X^{G_r} = X^{T_I}$ .

When  $H^{odd}(X(Q, v); \mathbb{Z}/p) = 0$ , Proposition 5.2 gives a constraint on the topology of  $Q_I$ , that is  $H_1(Q_I; \mathbb{Z}/p) = 0$ . It is proved in [14] that if  $X(Q, v)$  is a manifold and  $H^{odd}(X(Q, v); \mathbb{Z}) = 0$ , then  $Q$  is face-acyclic. This implies that there will be more constraints on the topology of  $Q_I$  when  $H^{odd}(X(Q, v); \mathbb{Z}/p) = 0$ , to be more precise, we expect that  $Q$  is *face  $p$ -acyclic* which means that (every component of)  $Q_I$  is acyclic with  $\mathbb{Z}/p$ -coefficients for every  $I$ . Therefore, in order to consider the converse of Proposition 5.2, it would be appropriate to assume that  $Q$  is face  $p$ -acyclic. We will prove in Section 7 that the converse holds in some cases while we will see in Section 8 that the converse does not hold in general.

## 6. THEOREM ON ELEMENTARY DIVISORS

We recall the Theorem on Elementary Divisors and deduce two facts from it, which will play a role in the next section.

**Theorem 6.1** (Theorem on Elementary Divisors, see [16]). *Let  $N'$  be a submodule of rank  $n'$  in  $N = \mathbb{Z}^n$ . Then there are bases  $\{u'_1, \dots, u'_{n'}\}$  of  $N'$  and  $\{u_1, \dots, u_n\}$  of  $N$  such that  $u'_i = \epsilon_i u_i$  with some integer  $\epsilon_i$  for  $i = 1, 2, \dots, n'$  and  $\epsilon_1 | \epsilon_2 | \dots | \epsilon_{n'}$ . Moreover if  $A = (a_1, \dots, a_k)$  is an  $n \times k$  integer matrix whose column vectors  $a_1, \dots, a_k$  generate  $N'$  and*

$$\delta_i := \gcd\{\det B \mid B \text{ is an } i \times i \text{ submatrix of } A\},$$

*then  $\delta_i = \delta_{i-1} \epsilon_i$  for  $i = 1, 2, \dots, n'$ . In particular, if  $n' = n$ , then  $\delta_n = |N/N'|$ .*

We deduce two facts from Theorem 6.1.

**Lemma 6.2.** *Let  $A$  be an  $n \times n$  integer matrix of rank  $n$  and  $\tilde{A}: \mathbb{R}^n/\mathbb{Z}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n$  be the epimorphism induced from  $A$ . Then  $\ker \tilde{A} \cong \text{coker } A$ .*

*Proof.* By Theorem 6.1 we may think of  $A$  as the diagonal matrix with diagonal entries  $\epsilon_1, \dots, \epsilon_n$ . Then one easily sees that  $\ker \tilde{A}$  and  $\text{coker } A$  are both isomorphic to  $\prod_{i=1}^n \mathbb{Z}/\epsilon_i$ , proving the lemma.  $\square$

Let  $a_1, \dots, a_{n+1}$  be elements of  $\mathbb{Z}^n$  which generate a sublattice  $\langle a_1, \dots, a_{n+1} \rangle$  of rank  $n$  and set  $d_i := |\det((a_j)_{j \neq i})|$  for  $i \in [n+1]$ . It follows from Theorem 6.1 that

$$(6.1) \quad \delta_n = \gcd(d_1, \dots, d_{n+1}) = |\mathbb{Z}^n / \langle a_1, \dots, a_{n+1} \rangle|.$$

Suppose that  $a_{n+1}$  is primitive. Let  $\bar{a}_k$  ( $k \neq n+1$ ) be the projection image of  $a_k$  on  $\mathbb{Z}^n / \langle a_{n+1} \rangle$  and let  $a'_k$  be the primitive vector in the quotient lattice  $\mathbb{Z}^n / \langle a_{n+1} \rangle$  which has the same direction as  $\bar{a}_k$  when  $\bar{a}_k$  is nonzero and  $a'_k$  be the zero vector when so is  $\bar{a}_k$ . Set  $d'_j := \det(a'_1, \dots, \widehat{a'_j}, \dots, a'_n)$ . With this understood we have the following.

**Lemma 6.3.**  $\gcd(d_1, \dots, d_n) \mid d_{n+1}$ , i.e.,  $\gcd(d_1, \dots, d_n) = \gcd(d_1, \dots, d_{n+1})$ . Moreover,  $\gcd(d'_1, \dots, d'_n) \mid \gcd(d_1, \dots, d_{n+1})$ .

*Proof.* Theorem 6.1 applied with  $N'$  generated by  $a_{n+1}$  says that there is a basis  $\{u_1, \dots, u_n\}$  of  $N = \mathbb{Z}^n$  such that  $a_{n+1} = \epsilon_1 u_1$  with some integer  $\epsilon_1$ . But since  $a_{n+1}$  is primitive, we have  $\epsilon_1 = \pm 1$ . Therefore, we may assume that  $a_{n+1} = (0, \dots, 0, 1)^T$  through a linear transformation of  $\mathbb{Z}^n$ . We have

$$(6.2) \quad d_{n+1} = |\det(a_1, \dots, a_n)| = \left| \sum_{j=1}^n a_j^n \tilde{a}_j^n \right|$$

where  $a_j^n$  is the  $(n, j)$  entry of the matrix  $(a_1, \dots, a_n)$  and  $\tilde{a}_j^n$  is its cofactor. Since  $a_{n+1} = (0, \dots, 0, 1)^T$ ,  $\tilde{a}_j^n$  agrees with  $d_j = |\det(a_1, \dots, \widehat{a_j}, \dots, a_{n+1})|$  up to sign.

Therefore  $\tilde{a}_j^n$  is divisible by  $\gcd(d_1, \dots, d_n)$  for every  $j$  and this together with (6.2) implies the former statement in the lemma.

Since  $a_{n+1} = (0, \dots, 0, 1)^T$ ,  $\mathbb{Z}^n / \langle a_{n+1} \rangle$  can naturally be identified with  $\mathbb{Z}^{n-1}$  and we have

$$(6.3) \quad d_j = |\det(a_1, \dots, \hat{a}_j, \dots, a_{n+1})| = |\det(\bar{a}_1, \dots, \hat{\bar{a}}_j, \dots, \bar{a}_n)| \quad \text{for } j = 1, 2, \dots, n$$

where  $\bar{a}_k$  ( $k = 1, 2, \dots, n$ ) is the projection image of  $a_k$  on  $\mathbb{Z}^n / \langle a_{n+1} \rangle = \mathbb{Z}^{n-1}$ . Since  $\bar{a}_k$  is a positive scalar multiple of  $a'_k$ ,  $d'_j = |\det(a'_1, \dots, \hat{a}'_j, \dots, a'_n)|$  divides the latter term in (6.3) above and hence  $d_j$ . This together with the former statement in the lemma implies the latter statement in the lemma.  $\square$

## 7. CONVERSE OF PROPOSITION 5.2 IN THREE CASES

In this section we show that if  $Q$  is face  $p$ -acyclic and has the same face poset as one of the following:

- Case 1:** the suspension  $\diamond^n$  of an  $(n-1)$ -simplex  $\Delta^{n-1}$  (see the Introduction),
- Case 2:** the  $n$ -simplex  $\Delta^n$ ,
- Case 3:** the prism  $\Delta^{n-1} \times [-1, 1]$ ,

then the converse of Proposition 5.2 holds, i.e. if  $\mu(Q_I)$  is finite and coprime to  $p$  for every  $I$ , then  $H^{\text{odd}}(X(Q, v); \mathbb{Z}/p) = 0$ .

First we establish Case 1. Then we reduce Case 2 to Case 1 by collapsing a face of  $Q$  to a point. In Case 3, according to the characteristic function  $v$ , we collapse one or two faces of  $Q$  to a point reducing Case 3 to Case 2 or Case 1. The argument then becomes much more complicated than that reducing Case 2 to Case 1. It would be interesting to see whether this inductive argument works for an arbitrary product of simplices.

Let  $q$  be a vertex of  $Q$ . Then  $q$  lies in  $Q_I$  for some  $I \subset [m]$  with  $|I| = n$ . We set

$$d_Q(q) := |\det((v_i)_{i \in I})|$$

where  $v_i = v(Q_i)$  as before.

**Case 1.** In this case  $Q$  has two vertices, say  $q$  and  $q'$ , and  $d_Q(q) = d_Q(q') = \mu(Q)$ .

**Proposition 7.1.** *Suppose that  $Q$  is face  $p$ -acyclic, has the same face poset as  $\diamond^n$  and  $\mu(Q)$  is coprime to  $p$ . Then  $X(Q, v)$  has the same cohomology as  $S^{2n}$  with  $\mathbb{Z}/p$ -coefficients, in particular  $H^{\text{odd}}(X(Q, v); \mathbb{Z}/p) = 0$ .*

*Proof.* When  $n = 1$ ,  $Q$  is a closed interval and  $X(Q, v)$  is homeomorphic to  $S^2$ ; so the proposition holds when  $n = 1$ . In the following we assume  $n \geq 2$ , so that  $Q$  has  $n$  facets.

Let  $T^n = (S^1)^n$ . Then  $\text{Hom}(S^1, T^n)$  is naturally isomorphic to  $\mathbb{Z}^n$  and we identify them. Let  $\{e_i\}_{i=1}^n$  be the standard basis of  $\mathbb{Z}^n$  and  $e: \{Q_1, \dots, Q_n\} \rightarrow \mathbb{Z}^n = \text{Hom}(S^1, T^n)$  be the characteristic function assigning  $e_i$  to  $Q_i$ . Then we have a  $T^n$ -space  $X(Q, e)$  which is actually a manifold because  $\{e_i\}_{i=1}^n$  is a basis of  $\mathbb{Z}^n$ .

The characteristic vectors  $v_i \in N = \text{Hom}(S^1, T)$  define an epimorphism  $\tilde{v}: T^n \rightarrow T$  sending  $(h_1, \dots, h_n)$  to  $\prod_{i=1}^n v_i(h_i)$ . One can see that the surjective map from  $Q \times T^n$  to  $Q \times T$  sending  $(q, t)$  to  $(q, \tilde{v}(t))$  descends to a  $\tilde{v}$ -equivariant map from  $X(Q, e)$  to  $X(Q, v)$  and further descends to a homeomorphism

$$X(Q, e) / \ker \tilde{v} \approx X(Q, v).$$

Here  $|\ker \tilde{v}| = |N/\hat{N}|$  by Lemma 6.2 and it is coprime to  $p$  by assumption. Moreover, since  $\ker \tilde{v}$  is a subgroup of the connected group  $T^n$  acting on  $X(Q, e)$ , the induced action of  $\ker \tilde{v}$  on  $H^*(X(Q, e); \mathbb{Z}/p)$  is trivial. Therefore we have

$$H^*(X(Q, e)/\ker \tilde{v}; \mathbb{Z}/p) \cong H^*(X(Q, e); \mathbb{Z}/p)$$

(see [2, Theorem 2.4 in p.120]) and hence it suffices to prove that  $X(Q, e)$  has the same cohomology as  $S^{2n}$  with  $\mathbb{Z}/p$ -coefficients.

Since  $Q$  has the same face poset as  $\diamond^n$  and every face of  $\diamond^n$  is contractible, there is a face preserving map  $f: Q \rightarrow \diamond^n$  which induces an isomorphism on the face posets. Since  $Q$  is face  $p$ -acyclic,  $f$  induces an isomorphism on cohomology with  $\mathbb{Z}/p$ -coefficients at each face. Similarly to the definition of  $e$ , one has a characteristic function on  $\diamond^n$ , also denoted by  $e$ . Then the map from  $Q \times T^n$  to  $\diamond^n \times T^n$  sending  $(q, t)$  to  $(f(q), t)$  descends to a map

$$X(Q, e) \rightarrow X(\diamond^n, e)$$

which induces an isomorphism on cohomology with  $\mathbb{Z}/p$ -coefficients. Since  $X(\diamond^n, e)$  is homeomorphic to  $S^{2n}$ , this proves the desired result.  $\square$

**Case 2.** Since  $Q$  has the same face poset as the  $n$ -simplex  $\Delta^n$ ,  $Q$  has  $n+1$  facets  $Q_1, \dots, Q_{n+1}$  and  $n+1$  vertices  $q_1, \dots, q_{n+1}$ . We number them in such a way that  $q_i$  is the unique vertex not contained in  $Q_i$ . It follows from (6.1) and Lemma 6.3 that

(7.1)

$$\begin{aligned} \mu(Q) &= \gcd(d_Q(q_1), \dots, d_Q(q_{n+1})) = \gcd(d_Q(q_1), \dots, \widehat{d_Q(q_i)}, \dots, d_Q(q_{n+1})) \text{ and} \\ \mu(Q_i) &\text{ divides } \mu(Q) \text{ for any } i \in [n+1]. \end{aligned}$$

In fact, the former identity in (7.1) follows from (6.1). The latter identity with  $i = n+1$  follows from Lemma 6.3 but the same proof of Lemma 6.3 works for any  $i$  and proves the desired identity. Similarly, the last assertion in (7.1) also follows from (the proof of) Lemma 6.3.

**Proposition 7.2.** *Suppose that  $Q$  is face  $p$ -acyclic, has the same face poset as  $\Delta^n$  and  $\mu(Q)$  is coprime to  $p$ . Then  $H^{\text{odd}}(X(Q, v); \mathbb{Z}/p) = 0$ .*

*Proof.* We abbreviate  $X(Q, v)$  as  $X$ . We prove the proposition by induction on  $n$ . When  $n = 1$ ,  $Q$  is a closed interval and  $X$  is homeomorphic to  $S^2$ ; so the proposition holds in this case. We assume that the proposition holds for any face  $p$ -acyclic  $(n-1)$ -dimensional manifold with corners satisfying the assumption in the proposition. For every  $i$ ,  $Q_i$  has the same face poset as  $\Delta^{n-1}$  and  $\mu(Q_i) | \mu(Q)$  by (7.1), so  $H^{\text{odd}}(X_i; \mathbb{Z}/p) = 0$  by the induction assumption, where  $X_i = \pi^{-1}(Q_i)$  and  $\pi: X \rightarrow Q$  is the quotient map. On the other hand, since  $\mu(Q) = \gcd(d_Q(q_1), \dots, d_Q(q_{n+1}))$  is coprime to  $p$  by assumption,  $d_Q(q_i)$  is coprime to  $p$  for some  $i$ . For such  $i$ ,  $Q/Q_i$  is face  $p$ -acyclic, has the same face poset as  $\diamond^n$  and  $\mu(Q/Q_i) = d_Q(q_i)$  is coprime to  $p$ , so  $H^{\text{odd}}(X/X_i; \mathbb{Z}/p) = 0$  by Proposition 7.1. These together with the exact sequence

$$\rightarrow H^{\text{odd}}(X/X_i; \mathbb{Z}/p) \rightarrow H^{\text{odd}}(X; \mathbb{Z}/p) \rightarrow H^{\text{odd}}(X_i; \mathbb{Z}/p) \rightarrow$$

show  $H^{\text{odd}}(X; \mathbb{Z}/p) = 0$ .  $\square$

**Case 3.** We denote the facets of  $Q$  corresponding to  $\Delta^{n-1} \times \{\pm 1\}$  by  $Q_{\pm}$  and the others by  $Q_1, \dots, Q_n$ . Accordingly, we abbreviate the characteristic vectors  $v(Q_{\pm})$  as

$v_{\pm}$  and  $v(Q_i)$  as  $v_i$ . We denote the vertices in  $Q_{\epsilon}$  by  $q_1^{\epsilon}, \dots, q_n^{\epsilon}$  for  $\epsilon = \pm$  in such a way that  $q_i^{\epsilon}$  is not contained in  $Q_i$ .

**Lemma 7.3.** *Suppose that  $Q$  is face  $p$ -acyclic and has the same face poset as  $\Delta^{n-1} \times [-1, 1]$ . If  $\mu(Q)$  is coprime to  $p$  and either  $\mu(Q_+)$  or  $\mu(Q_-)$  is coprime to  $p$ , then there is a vertex  $q$  of  $Q$  such that  $d_Q(q)$  is coprime to  $p$ .*

We will prove this lemma later. It suffices to prove the following for our purpose in Case 3.

**Proposition 7.4.** *Suppose that  $Q$  is face  $p$ -acyclic, has the same face poset as  $\Delta^{n-1} \times [-1, 1]$  and  $\mu(Q)$ ,  $\mu(Q_{\pm})$  are coprime to  $p$ . Then  $H^{\text{odd}}(X(Q, v); \mathbb{Z}/p) = 0$ .*

*Proof.* We abbreviate  $X(Q, v)$  as  $X$  and denote by  $X_{\epsilon}$  ( $\epsilon = +$  or  $-$ ) the inverse image of  $Q_{\epsilon}$  by the quotient map  $\pi: X \rightarrow Q$ . Since  $Q_{\epsilon}$  is face  $p$ -acyclic, has the same face poset as  $\Delta^{n-1}$  and  $\mu(Q_{\epsilon})$  is coprime to  $p$  by assumption, we have

$$(7.2) \quad H^{\text{odd}}(X_{\epsilon}; \mathbb{Z}/p) = 0$$

by Proposition 7.2.

By Lemma 7.3 there is a vertex  $q$  of  $Q$  such that  $d_Q(q)$  is coprime to  $p$ . Without loss of generality we may assume  $q = q_n^-$ , i.e.  $d_Q(q_n^-)$  is coprime to  $p$ . Since we have (7.2) and the exact sequence

$$\rightarrow H^{\text{odd}}(X/X_+; \mathbb{Z}/p) \rightarrow H^{\text{odd}}(X; \mathbb{Z}/p) \rightarrow H^{\text{odd}}(X_+; \mathbb{Z}/p) \rightarrow,$$

it suffices to prove

$$(7.3) \quad H^{\text{odd}}(X/X_+; \mathbb{Z}/p) = 0.$$

We consider two cases.

*Case a.* The case where  $\det(v_1, \dots, v_n) \neq 0$ . In this case, the characteristic function  $v$  on  $Q$  induces a characteristic function on  $Q/Q_+$ , denoted  $v^+$ , and  $X/X_+ = X(Q/Q_+, v^+)$ . We note that  $Q/Q_+$  is face  $p$ -acyclic and has the same face poset as  $\Delta^n$  since  $Q$  is face  $p$ -acyclic and has the same poset as  $\Delta^{n-1} \times [-1, 1]$ . Moreover, since  $q_n^-$  is a vertex of  $Q/Q_+$  and  $d_{Q/Q_+}(q_n^-) = d_Q(q_n^-)$  is coprime to  $p$ ,  $\mu(Q/Q_+)$  is coprime to  $p$ . Therefore, (7.3) follows from Proposition 7.2.

*Case b.* The case where  $\det(v_1, \dots, v_n) = 0$ .

**Claim.** There is a vertex  $q$  of  $Q_n$  such that  $d_{Q_n}(q)$  is coprime to  $p$ , so  $\mu(Q_n)$  is coprime to  $p$ .

*Proof.* Write  $v_i = (v_i^1, \dots, v_i^n)^T$  and  $v_- = (v_-^1, \dots, v_-^n)^T$ . Since  $v_n$  is primitive, we may assume  $v_n = (0, \dots, 0, 1)^T$  by Theorem 6.1. Denote by  $\bar{v}_i$  and  $\bar{v}_-$  the projection images of  $v_i$  and  $v_-$  on  $\mathbb{Z}^n / \langle v_n \rangle$  and by  $v'_i$  and  $v'_-$  the primitive vectors which have the same directions as  $\bar{v}_i$  and  $\bar{v}_-$  respectively. Then

$$d_{Q_n}(q_i^-) = |\det(v'_1, \dots, \widehat{v'_i}, \dots, v'_{n-1}, v'_-)|$$

by definition and hence

$$(7.4) \quad d_{Q_n}(q_i^-) \mid \det(\bar{v}_1, \dots, \widehat{\bar{v}_i}, \dots, \bar{v}_{n-1}, \bar{v}_-).$$

On the other hand, since  $v_n = (0, \dots, 0, 1)^T$ , we have

$$\det(v_1, \dots, v_n) = \det(\bar{v}_1, \dots, \bar{v}_{n-1})$$

and the left hand side above is zero by assumption. It follows that

$$\begin{aligned}
d_Q(q_n^-) &= |\det(v_1, \dots, v_{n-1}, v_-)| \\
&= |v_-^n \det(\bar{v}_1, \dots, \bar{v}_{n-1}) + \sum_{j=1}^{n-1} v_j^n (-1)^{n-j} \det(\bar{v}_1, \dots, \hat{\bar{v}}_j, \dots, \bar{v}_{n-1}, \bar{v}_-)| \\
&= |\sum_{j=1}^{n-1} v_j^n (-1)^{n-j} \det(\bar{v}_1, \dots, \hat{\bar{v}}_j, \dots, \bar{v}_{n-1}, \bar{v}_-)|
\end{aligned}$$

where the second identity above is the expansion of  $\det(v_1, \dots, v_{n-1}, v_-)$  with respect to the  $n$ th row. By (7.4)  $\gcd(d_{Q_n}(q_1^-), \dots, d_{Q_n}(q_{n-1}^-))$  divides the last term above. Since  $d_Q(q_n^-)$  is coprime to  $p$ , this means that  $d_{Q_n}(q_i^-)$  is coprime to  $p$  for some  $i$ , proving the claim.

Now we shall prove (7.3) by induction on the dimension  $n$  of  $Q$ . When  $n = 1$ ,  $Q$  is a closed interval,  $X$  is  $S^2$  and  $X_+$  is a point; so (7.3) holds in this case. We assume  $n \geq 2$  in the following. Let  $X_n$  be the inverse image of  $Q_n$  by the quotient map  $\pi: X \rightarrow Q$ . The face poset of  $Q_n$  is the same as that of  $\Delta^{n-2} \times [-1, 1]$  and  $Q_n$  is face  $p$ -acyclic. The facets corresponding to  $\Delta^{n-2} \times \{\pm 1\}$  are  $Q_n \cap Q_\pm$  and  $\mu(Q_n \cap Q_\pm)$  are coprime to  $p$  by (7.1) because  $\mu(Q_\pm)$  are coprime to  $p$  by assumption. Moreover,  $\mu(Q_n)$  is also coprime to  $p$  by the claim above. Therefore

$$(7.5) \quad H^{odd}(X_n/(X_n \cap X_+); \mathbb{Z}/p) = 0$$

by the induction assumption.

The quotient  $Q/(Q_n \cup Q_+) =: \tilde{Q}$  is face  $p$ -acyclic and  $\tilde{Q}$  has the same face poset as  $\diamond^n$ . The characteristic function  $v$  on  $Q$  induces a characteristic function on  $\tilde{Q}$ , denoted  $\tilde{v}$ , because  $q_n^-$  is a vertex of  $\tilde{Q}$  and  $d_{\tilde{Q}}(q_n^-) = d_Q(q_n^-)$  is coprime to  $p$ , in particular nonzero. The quotient space  $X_n/(X_n \cap X_+)$  is a subspace of  $X/X_+$  and

$$(7.6) \quad (X/X_+) / (X_n/(X_n \cap X_+)) = X(\tilde{Q}, \tilde{v}).$$

Since  $d_{\tilde{Q}}(q_n^-) = \mu(\tilde{Q})$  is coprime to  $p$ ,  $H^{odd}(X(\tilde{Q}, \tilde{v}); \mathbb{Z}/p) = 0$  by Proposition 7.1. This together with (7.6), (7.5) and the exact sequence

$$\begin{aligned}
&\rightarrow H^{odd}((X/X_+) / (X_n/(X_n \cap X_+)); \mathbb{Z}/p) \rightarrow H^{odd}(X/X_+; \mathbb{Z}/p) \\
&\rightarrow H^{odd}(X_n/(X_n \cap X_+); \mathbb{Z}/p) \rightarrow
\end{aligned}$$

implies (7.3).  $\square$

Now it remains to prove Lemma 7.3.

*Proof of Lemma 7.3.* We may assume that  $\mu(Q_+)$  is coprime to  $p$ . We may also assume that  $v_+ = (0, \dots, 0, 1)^T$  by Theorem 6.1 through some identification of  $N$  with  $\mathbb{Z}^n$ . Suppose that

$$(7.7) \quad p \mid d_Q(q) \text{ for all vertices } q \text{ of } Q$$

and we will deduce a contradiction in the following.

By Lemma 6.3,  $\det(v_1, \dots, v_n)$  is divisible by  $\gcd(d_Q(q_1^\epsilon), \dots, d_Q(q_n^\epsilon))$ , so it follows from (7.7) that

$$(7.8) \quad p \mid \det(v_1, \dots, v_n).$$

We write  $v_i = (v_i^1, \dots, v_i^n)^T \in \mathbb{Z}^n$  for  $i = 1, 2, \dots, n$ .



**Claim 1.** There is an  $i \in [n]$  such that  $p \mid v_i^j$  for all  $j \neq n$ .

*Proof.* Since  $v_+ = (0, \dots, 0, 1)^T$ , we naturally identify the quotient lattice  $\mathbb{Z}^n / \langle v_+ \rangle$  with  $\mathbb{Z}^{n-1}$  and then the projection image  $\bar{v}_i$  of  $v_i$  on the quotient lattice  $\mathbb{Z}^{n-1}$  is  $(v_i^1, \dots, v_i^{n-1})$ . Set  $s_i = \gcd(v_i^1, \dots, v_i^{n-1})$ . Then  $\bar{v}_i/s_i =: v'_i$  is primitive. Since  $d_Q(q)$  is assumed to be divisible by  $p$  for all vertices  $q$  of  $Q$ , we have

$$(7.9) \quad p \mid \det(v_{i_1}, \dots, v_{i_{n-1}}, v_+) \quad \text{for every subset } \{i_1, \dots, i_{n-1}\} \text{ of } [n].$$

Here, since  $v_+ = (0, \dots, 0, 1)^T$ , we have

$$(7.10) \quad \det(v_{i_1}, \dots, v_{i_{n-1}}, v_+) = \det(\bar{v}_{i_1}, \dots, \bar{v}_{i_{n-1}}) = \left( \prod_{k=1}^{n-1} s_{i_k} \right) \det(v'_{i_1}, \dots, v'_{i_{n-1}}).$$

Now suppose that  $s_i$  is not divisible by  $p$  for any  $i$ . Then it follows from (7.9) and (7.10) that  $p \mid \det(v'_{i_1}, \dots, v'_{i_{n-1}})$  for every subset  $\{i_1, \dots, i_{n-1}\}$  of  $[n]$ . Since  $\mu(Q_+)$  agrees with the greatest common divisor of all  $\det(v'_{i_1}, \dots, v'_{i_{n-1}})$  by (6.1), this shows that  $p \mid \mu(Q_+)$  which contradicts the assumption that  $\mu(Q_+)$  is coprime to  $p$ . Therefore  $p \mid s_i$  for some  $i$ , proving the claim.

**Claim 2.**  $p \mid \det(v_{i_1}, \dots, v_{i_{n-2}}, v_-, v_+)$  for every subset  $\{i_1, \dots, i_{n-2}\}$  of  $[n]$ .

*Proof.* Since  $v_+ = (0, \dots, 0, 1)^T$ , we have

$$(7.11) \quad \det(v_{i_1}, \dots, v_{i_{n-2}}, v_-, v_+) = \det(\bar{v}_{i_1}, \dots, \bar{v}_{i_{n-2}}, \bar{v}_-)$$

where  $\bar{v}_- = (v_-^1, \dots, v_-^{n-1})^T$  is the projection image of  $v_-$  on the quotient  $\mathbb{Z}^n / \langle v_+ \rangle = \mathbb{Z}^{n-1}$ . We shall observe that the right hand side in (7.11) is divisible by  $p$ . Without loss of generality we may assume that the  $i$  in Claim 1 is  $n$ , so that  $p \mid v_n^j$  for all  $j \neq n$ . We consider two cases.

*Case a.* The case where  $n \in \{i_1, \dots, i_{n-2}\}$ . Since  $\bar{v}_n = (v_n^1, \dots, v_n^{n-1})^T$  and  $p \mid v_n^j$  for all  $j \neq n$ , the right hand side in (7.11) is divisible by  $p$ .

*Case b.* The case where  $n \notin \{i_1, \dots, i_{n-2}\}$ . In this case, we consider the expansion of  $\det(v_{i_1}, \dots, v_{i_{n-2}}, v_-, v_n)$  with respect to the last column. Since  $v_n = (v_n^1, \dots, v_n^n)^T$  and  $p \mid v_n^j$  for all  $j \neq n$ , we have

$$(7.12) \quad |\det(v_{i_1}, \dots, v_{i_{n-2}}, v_-, v_n)| \equiv |v_n^n \det(\bar{v}_{i_1}, \dots, \bar{v}_{i_{n-2}}, \bar{v}_-)| \pmod{p}.$$

Here the left hand side above is  $d_Q(q)$  for  $q = (\bigcap_{k=1}^{n-2} Q_{i_k}) \cap Q_- \cap Q_n$ , so it is divisible by  $p$  by (7.7). Moreover,  $v_n^n$  is not divisible by  $p$  because otherwise every entry of  $v_n$  is divisible by  $p$  and this contradicts  $v_n$  being primitive. It follows from (7.12) that the right hand side in (7.11) is divisible by  $p$  in this case, too.

This completes the proof of the claim.

Now (7.7), (7.8) and Claim 2 show that all  $n \times n$  minors of  $(v_1, \dots, v_n, v_-, v_+)$  are divisible by  $p$  and hence  $p \mid \mu(Q) (= |N/\hat{N}|)$  by Theorem 6.1. This contradicts the assumption that  $\mu(Q)$  is coprime to  $p$ , proving the lemma.  $\square$

## 8. EXAMPLE

In this section we shall give an example of a compact simplicial toric variety showing that the converse of Proposition 5.2 does not hold in general.

Let  $Q$  be the 3-dimensional simple polytope with the 7 facets  $Q_+, Q_-, Q_1, \dots, Q_5$ , where  $Q_4$  and  $Q_5$  are triangles obtained by cutting two vertices of a prism, shown in Figure 1 below. The polytope  $Q$  can be obtained from  $\diamond^3$  by performing a vertex cut four times.

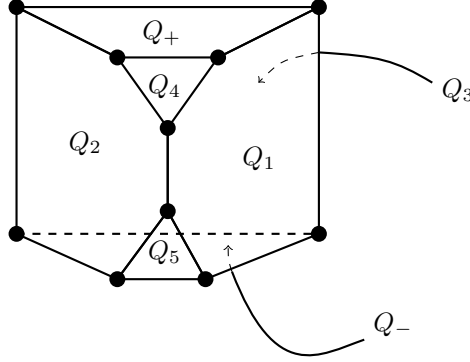


FIGURE 1.

Let  $d$  be a positive integer. To the 7 facets  $Q_1, \dots, Q_5, Q_+, Q_-$ , we respectively assign the following vectors

$$\begin{aligned} v_1 &= (1, 0, 0) & v_2 &= (-1, d, -d) & v_3 &= (-1, -d, 0) \\ v_4 &= (0, 1, 0) & v_5 &= (d, 1-d, -d) \\ v_+ &= (0, 0, 1) & v_- &= (1, -1, -1), \end{aligned}$$

giving a characteristic function  $v$  on  $Q$ . There are ten vertices in  $Q$ . At each vertex, there are exactly three facets meeting and the determinant of the three vectors assigned to the facets is nonzero, indeed their absolute values are as follows:

$$\begin{aligned} |\det(v_1, v_4, v_+)| &= |\det(v_2, v_4, v_+)| = |\det(v_1, v_5, v_-)| = 1 \\ |\det(v_1, v_2, v_4)| &= |\det(v_1, v_3, v_+)| = |\det(v_1, v_3, v_-)| = d \\ |\det(v_1, v_2, v_5)| &= d(2d-1) & |\det(v_2, v_5, v_-)| &= d+1 \\ |\det(v_2, v_3, v_-)| &= d(d+3) & |\det(v_2, v_3, v_+)| &= 2d. \end{aligned}$$

(Precisely speaking, the vectors are regarded as column vectors here by taking transpose.) Therefore, at each vertex, the cone spanned by the three vectors is 3-dimensional and has the origin as the apex. One can also check that

$$\begin{aligned} v_4 &= (v_1 + v_2 + dv_+)/d & v_5 &= ((d+1)v_1 + v_2 + d(2d-1)v_-)/2d \\ v_+ &= -(2v_1 + v_2 + v_3)/d & v_- &= ((d+3)v_1 + v_2 + 2v_3)/d. \end{aligned}$$

Since  $d$  is a positive integer, this shows that  $-v_+$  is in the cone  $\angle v_1 v_2 v_3$  and  $v_4$  is in the cone  $\angle v_1 v_2 v_+$  while  $v_-$  is in the cone  $\angle v_1 v_2 v_3$  and  $v_5$  is in the cone  $\angle v_1 v_2 v_-$  (see Figure 2), where  $\angle uvw$  denotes the cone spanned by vectors  $u, v, w$ . This implies that the ten 3-dimensional cones have no overlap and cover the entire  $\mathbb{R}^3$ , giving a complete simplicial fan so that the quotient space  $X = X(Q, v)$  is homeomorphic to a compact simplicial toric variety.

We shall check that  $\mu(Q_I) = 1$  for each face  $Q_I$  of  $Q$ , where  $\mu(Q_I)$  is defined in Section 5. As remarked in Section 5,  $\mu(Q_I) = 1$  when  $|I| = 2$  or  $3$ . Clearly  $\hat{N} = N(= \mathbb{Z}^3)$ . Therefore it suffices to check  $\mu(Q_I) = 1$  when  $|I| = 1$ . At vertices  $Q_1 \cap Q_4 \cap Q_+$ ,  $Q_2 \cap Q_4 \cap Q_+$  and  $Q_1 \cap Q_5 \cap Q_-$ , we have

$$|\det(v_1, v_4, v_+)| = |\det(v_2, v_4, v_+)| = |\det(v_1, v_5, v_-)| = 1$$

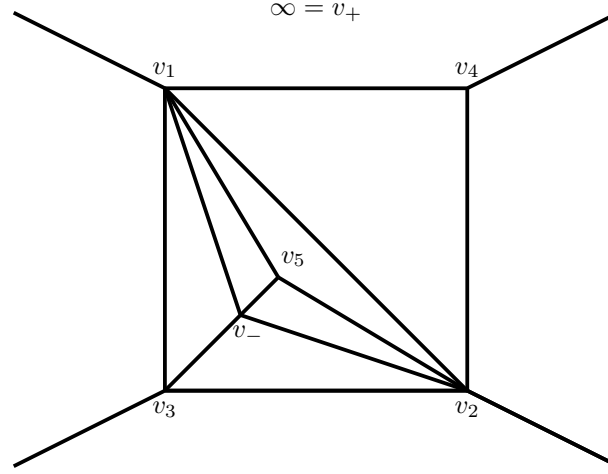


FIGURE 2. Each vector  $v_i$  is denoted by a point in  $\mathbb{R}^2 \cup \{\infty\}$  and a segment connecting  $v_i, v_j$  corresponds to the 2-dimensional cone spanned by them and a triangle formed by  $v_i, v_j, v_k$  corresponds to the 3-dimensional cone spanned by them.

and hence  $\mu(Q_I) = 1$  for every  $I$  with  $|I| = 1$  except  $I = \{3\}$  again by the remark in Section 5. In order to see  $\mu(Q_3) = 1$ , we note that  $\{v_3, v_4, v_+\}$  is a base of  $N$  and

$$v_1 = -v_3 - dv_4, \quad v_2 = v_3 + 2dv_4 - dv_+.$$

Therefore, the images of  $v_1$  and  $v_2$  by the quotient map  $\pi_{\{3\}}: N \rightarrow N(\{3\}) = N/\langle v_3 \rangle$  are  $(-d, 0)$  and  $(2d, -d)$  with respect to the base  $\{\pi_{\{3\}}(v_4), \pi_{\{3\}}(v_+)\}$ . Thus the corresponding primitive vectors are  $(-1, 0)$  and  $(2, -1)$  which form a base of  $N(\{3\})$ . Hence  $\mu(Q_3) = 1$ .

We shall compute  $H^3(X)$ . Take a plane in  $\mathbb{R}^3$  which meets the facets  $Q_1, Q_2, Q_3$  transversally and does not meet the other facets of  $Q$ . Cutting  $Q$  along the plane, we divide  $Q$  into two polytopes, denoted  $P_+$  and  $P_-$  containing  $Q_+$  and  $Q_-$  respectively. Let  $\pi: X \rightarrow Q$  be the quotient map and set

$$Y_\epsilon := \pi^{-1}(P_\epsilon) \text{ for } \epsilon = \pm, \quad Y := Y_+ \cap Y_-, \quad P := P_+ \cap P_-.$$

The quotient space  $P_\epsilon/P$  can be regarded as a prism. The characteristic function  $v$  on  $Q$  induces a characteristic function on  $P_\epsilon/P$ , denoted  $w_\epsilon$ , and  $X/Y_+ = Y_-/Y$  (resp.  $X/Y_- = Y_+/Y$ ) is homeomorphic to  $X(P_-/P, w_-)$  (resp.  $X(P_+/P, w_+)$ ). The same argument as above shows that  $\mu$  takes 1 on all faces of the prism  $P_\epsilon/P$ , so

$$(8.1) \quad H^*(X, Y_\epsilon) \text{ and } H^*(Y_\epsilon, Y) \text{ are torsion free and vanish in odd degrees}$$

by Proposition 7.4.

Let  $\tilde{Q}$  be a nice manifold with corners obtained from  $Q$  by collapsing  $Q_4 \cup Q_+$  and  $Q_5 \cup Q_-$  to a point respectively. The  $\tilde{Q}$  has three facets coming from  $Q_1, Q_2, Q_3$  and the characteristic function  $v$  on  $Q$  induces a characteristic function  $\tilde{v}$  on  $\tilde{Q}$ . Since

$$v_1 = (1, 0, 0), \quad v_2 = (-1, d, -d), \quad v_3 = (-1, -d, 0),$$

one can see that  $H^4(X(\tilde{Q}, \tilde{v})) \cong \mathbb{Z}/d$  by Corollary 4.2, and since  $X(\tilde{Q}, \tilde{v})$  is homeomorphic to the suspension of  $Y$ , we obtain

$$(8.2) \quad H^3(Y) \cong \mathbb{Z}/d.$$

Now, consider the exact sequence in cohomology for the pair  $(Y_+, Y)$ :

$$(8.3) \quad \rightarrow H^3(Y_+, Y) \rightarrow H^3(Y_+) \rightarrow H^3(Y) \rightarrow H^4(Y_+, Y) \rightarrow .$$

Since  $H^3(Y_+, Y) = 0$  and  $H^4(Y_+, Y)$  is torsion free by (8.1) and  $H^3(Y)$  is a torsion group by (8.2), it follows from the exact sequence (8.3) that

$$(8.4) \quad H^3(Y_+) \cong H^3(Y) \cong \mathbb{Z}/d.$$

Next, consider the exact sequence in cohomology for the pair  $(X, Y_+)$ :

$$(8.5) \quad \rightarrow H^3(X, Y_+) \rightarrow H^3(X) \rightarrow H^3(Y_+) \rightarrow H^4(X, Y_+) \rightarrow .$$

Similarly to the above argument,  $H^3(X, Y_+) = 0$  and  $H^4(X, Y_+)$  is torsion free by (8.1) and  $H^3(Y_+)$  is a torsion group by (8.4), so it follows from the exact sequence (8.5) that

$$H^3(X) \cong H^3(Y_+) \cong \mathbb{Z}/d.$$

Thus  $X = X(Q, v)$  is the desired example when  $d \geq 2$ .

#### APPENDIX

In this appendix, we observe that when  $X$  is a compact simplicial toric variety of complex dimension  $n$ , a result of Fischli [7] or Jordan [11] implies that  $H^{2n-1}(X) \cong N/\hat{N}$  and  $\text{Tor } H^{2n-2}(X) \cong \wedge^2 N/(\hat{N} \wedge N)$ , where  $\text{Tor } H^{2n-2}(X)$  denotes the torsion part of  $H^{2n-2}(X)$ . This result agrees with Proposition 2.2 since  $Q$  is contractible in this case.

Let  $\Delta$  be a simplicial complete fan of dimension  $n$  and let  $X$  be the associated compact simplicial toric variety. Let  $M$  be the free abelian group dual to  $N$ . Since  $N = \text{Hom}(S^1, T)$ ,  $M$  can be thought of as  $\text{Hom}(T, S^1)$ . According to [7, Theorem 2.3] or [11, Theorem 2.5.5],

$$H^{2n-1}(X) \cong \text{coker } \delta_1, \quad \text{Tor } H^{2n-2}(X) \cong \text{coker } \delta_2,$$

where

$$(8.6) \quad \delta_r: \bigoplus_{\tau \in \Delta^{(1)}} \wedge^{n-r}(\tau^\perp \cap M) \rightarrow \wedge^{n-r} M \quad (r = 1, 2)$$

is the sum of inclusion maps with signs,  $\Delta^{(1)}$  denotes the set of one-dimensional cones in  $\Delta$  and  $\tau^\perp$  denotes the subspace of  $M \otimes \mathbb{R}$  which vanish on  $\tau$ .

We shall interpret the above in terms of  $N$ . Let  $\sigma$  be a cone of dimension  $n - k$  in  $\Delta$ . Then we have

$$(8.7) \quad \begin{aligned} \wedge^\ell(\sigma^\perp \cap M) &\cong \text{Hom}(\wedge^{k-\ell}(\sigma^\perp \cap M), \mathbb{Z}) \quad (\because \text{rank } \sigma^\perp \cap M = k) \\ &\cong \wedge^{k-\ell}(N/N_\sigma) \quad (\because N/N_\sigma \text{ is dual to } \sigma^\perp \cap M) \\ &\cong (\wedge^{n-k} N_\sigma) \wedge (\wedge^{k-\ell} N) \end{aligned}$$

where  $N_\sigma$  is the intersection of  $N$  with the subspace of  $N \otimes \mathbb{R}$  spanned by  $\sigma$ . The last isomorphism above is given as follows. Choose a base  $\rho_1, \dots, \rho_{n-k}$  of  $N_\sigma$ . Since  $N_\sigma$  is of rank  $n - k$ ,  $\wedge^{n-k} N_\sigma$  is a free abelian group of rank one and  $\rho_1 \wedge \dots \wedge \rho_{n-k}$

is its generator. For  $w \in N$ , we denote by  $[w]$  the element of  $N/N_\sigma$  determined by  $w$ . Then the following correspondence

$$[w_1] \wedge \cdots \wedge [w_{k-\ell}] \rightarrow \rho_1 \wedge \cdots \wedge \rho_{n-k} \wedge w_1 \wedge \cdots \wedge w_{k-\ell}$$

is well defined and gives the desired isomorphism from  $\wedge^{k-\ell}(N/N_\sigma)$  to  $(\wedge^{n-k}N_\sigma) \wedge (\wedge^{k-\ell}N)$ . This isomorphism is independent of the choice of the base  $\rho_1, \dots, \rho_{n-k}$  up to sign. Namely, the isomorphism (8.7) depends only on the choice of orientations on  $M$  (or  $N$ ) and  $\sigma$ .

Applying (8.7) to  $\sigma = \tau \in \Delta^{(1)}$  and  $\sigma = 0$ , we obtain

$$\begin{aligned} \wedge^{n-1}(\tau^\perp \cap M) &\cong N_\tau, & \wedge^{n-1}M &\cong N, \\ \wedge^{n-2}(\tau^\perp \cap M) &\cong N_\tau \wedge N, & \wedge^{n-2}M &\cong \wedge^2 N. \end{aligned}$$

Since  $\delta_r$  is the sum of inclusion maps with signs, the image of  $\delta_1$  (resp.  $\delta_2$ ) in (8.6) can be identified with  $\hat{N}$  (resp.  $\hat{N} \wedge N$ ) and hence

$$H^{2n-1}(X) \cong E_2^{n,n-1} \cong N/\hat{N}, \quad \text{Tor } H^{2n-2}(X) \cong E_2^{n,n-2} \cong \wedge^2 N/(\hat{N} \wedge N).$$

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